Section 5.5

Indefinite Integrals

We continue to expand our integration toolbox.

Example 5.5.1. $\int \cos 2x \, dx$

Example 5.5.2. $\int \cos(2x + 5) \, dx$

Why did the above work? What did the two indefinite integrals have in common?

Theorem 5.5.3 (Substitution Rule for Indefinite Integrals)**.** Let $u = g(x)$, where g is differentiable on an interval and f be continuous on the corresponding range of g. On that same interval

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

This is particularly helpful if $F(x)$ where $f(x) = F'(x)$ is known, as the integral becomes $F(g(x)) + C$.

Example 5.5.4. Find the indefinite integral:

1. $\int 2\big(2x+1\big)^3 dx$

2. $\int x^4\big(x^5+6\big)^9 dx$

Procedure 5.5.5.

1. Given an indefinite integral of a composite function $f(g(x))$, identify the inner function $u = g(x)$ such that $g'(x)dx$ appears as a differential factor in the integral.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ in the integral.
3. Evaluate the indefinite integral in terms of u.
4. Write the result back in terms of x using the relation $u = g(x)$.

Disclaimer: Substitution will always produce an integral that is equivalent but will not necessarily be any more computable.

3. $\int \cos^3 x \sin x \, dx$

4. $\int 7^x dx$

5. $\int \frac{dx}{\sqrt{16 - 25x^2}}$

Theorem 5.5.6 (Useful Formulas)**.**

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos ax + C \qquad \int \cos(ax)\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + C \qquad \int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

$$\int b^x dx = \frac{1}{\ln b}b^x + C \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \qquad \int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\text{arcsec}\left|\frac{x}{a}\right| + C$$

Definite Integrals

In indefinite integrals we write both $g(x)$ and dx in terms of u and its differential; for definite integrals we also need to rewrite the limits of integration as $g(a)$ and $g(b)$.

Theorem 5.5.7 (Substitution Rule for Definite Integrals)**.** Let $u = g(x)$ where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 5.5.8. Compute the definite integral:

1. $\displaystyle\int_0^2 \frac{2x\,dx}{\left(x^2+2\right)^2}$

2. $\int_1^3 \frac{2^x}{2^x+4} dx$

3. $\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\, d\theta$

4. $\int_0^{\frac{1}{5}} \frac{dx}{\sqrt{1-25x^2}}$